Disc. sec. 7

nuts & bolts case study (continued)

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An IID random sample of $m = 114$ nut/bolt pairs is drawn; $n = 3$ were defective

Let θ = defective rate over a much longer sampling period (assume stationarity)

Reasonable intuitive estimate of θ is $\hat{\theta} = \frac{n}{m} = \frac{3}{114} \doteq 0.026$

Let

$$B_i = \begin{cases} 1 & \text{if sampled pair } i \text{ defective} \\ 0 & \text{else} \end{cases}$$

$(i = 1, \ldots, m)$; $(B_i \mid \theta) \stackrel{IID}{\sim} \text{Bernoulli}(\theta)$

$N = \sum_{i=1}^{m} B_i$ counts # of defectives in sample: $(N \mid \theta) \sim \text{Binomial}(m, \theta)$

~~pmf~~

$f_N(n \mid \theta) = P(N = n \mid \theta) =$

$$p(N = n \mid \theta) = \begin{cases} \binom{m}{n} \theta^n (1-\theta)^{m-n} & \text{for } n = 0, 1, \ldots, m \\ 0 & \text{else} \end{cases}$$

sampling dist.

$(0 < \theta < 1)$

$\ell(\theta \mid n) = c \, p(N = n \mid \theta)$

likelihood function for θ

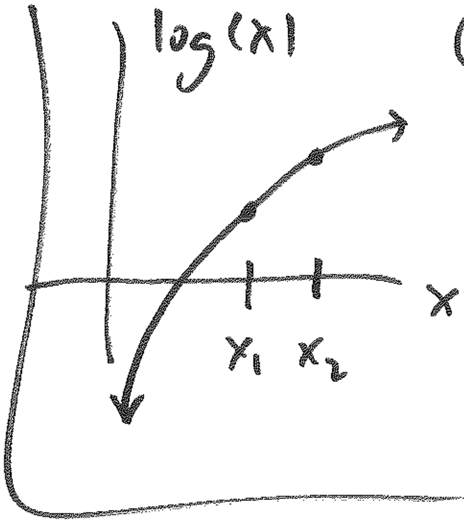
positive

$= c \binom{m}{n} \theta^n (1-\theta)^{m-n}$

$= c \, \theta^n (1-\theta)^{m-n}$

plot with $c = 1$, $n = 3$, $m = 114$

$$\ell(\theta|n) = \theta^{n}(1-\theta)^{m-n}$$



$\ell\ell(\theta|n)$ log likelihood f'n

$$\ell\ell(\theta|n) = n\log\theta + (m-n)\log(1-\theta)$$

$$\frac{d}{d\theta}\ell\ell(\theta|n) = \frac{n}{\theta} - \frac{m-n}{1-\theta}$$

$$= \frac{n - \cancel{n\theta} - m\theta + \cancel{n\theta}}{\theta(1-\theta)}$$

$$= \frac{n - m\theta}{\theta(1-\theta)}$$

So $\frac{d}{d\theta}\ell\ell(\theta|n) = 0$

iff $n - m\theta = 0$

i.e. iff $\theta = \hat{\theta}_{MLE} = \frac{n}{m}$

Bayes's Theorem

posterior dist. — prior dist. — likelihood dist.

$$\boxed{p(\theta|n)} = c\,\boxed{p(\theta)}\,\boxed{\ell(\theta|n)}$$

$$\boxed{\theta^{n}(1-\theta)^{m-n}}$$

$$\theta \sim \text{Beta}(\alpha, \beta) \longleftrightarrow \begin{pmatrix} \alpha > 0 \\ \beta > 0 \end{pmatrix}$$

$$p(\theta) = c \quad \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

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if we choose $p(\theta) = \text{Beta}(\alpha, \beta)$

then

$$p(\theta | n) = c \, \overbrace{\boxed{\theta^{\alpha-1} (1-\theta)^{\beta+1}}}^{\text{prior}} \; \overbrace{\theta^{n} (1-\theta)^{m-n}}^{\text{lik}}$$

$$= c \quad \theta^{(\alpha+n)-1} (1-\theta)^{(\beta+m-n)-1}$$

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in Binomial sampling model

$(N | \theta) \sim \text{Binomial}(m, \theta)$,

if $\theta \sim \text{Beta}(\alpha, \beta)$

then $(\theta | n) \sim \text{Beta}(\alpha+n, \beta+m-n)$

prior $\theta \sim \text{Uniform}(0,1)$ $\Leftrightarrow$ θ could be anywhere between 0 & 1, with no value particularly favored over any other value (diffuse prior)

$$p(\theta) = \begin{cases} 1 & 0 < \theta < 1 \\ 0 & \text{else} \end{cases}$$

$$\text{Uniform}(0,1) = \text{Beta}(1,1)$$