This conditional time: probability
Next discrete time: distributions

Village: Some peopleesty:
cause → effect

A

B_i:

\[ P(A|B_i) = P(\text{effect} | \text{cause}) \] easier

or \[ P(B_i|A) = P(\text{cause} | \text{effect}) \] harder but the goal

Bayes wanted to reverse order of conditioning:
cause \rightarrow effect

\textbf{Bi} \rightarrow A

\text{unknown} \rightarrow \text{data}

\text{data} \rightarrow \text{final information processing}

\text{p}(\text{cause} \mid \text{effect}) = \frac{\text{p}(\text{unknown} \mid \text{data}) \cdot \text{p}(\text{data} \mid \text{unknown})}{\text{p}(\text{data})}

\text{p}(\text{data} \mid \text{unknown}) = \frac{\text{p}(\text{data} \mid \text{unknown})}{\text{p}(\text{data})}

\text{p}(\text{data}) = \frac{\text{p}(\text{data} \mid \text{unknown})}{\text{p}(\text{unknown} \mid \text{data})}

\text{before data} \rightarrow \text{after data} \rightarrow \text{time} \rightarrow \text{prior info.} \rightarrow \text{data info} \rightarrow \text{posterior info.}

\text{data info} \rightarrow \text{posterior info.}

\text{posterior info} = \frac{\text{prior info} \cdot \text{data info} \cdot \text{likelihood}}{\text{annoying normalizing constant}}

\text{info. about unknown} \rightarrow \text{data queries}

\text{a priori}
\( P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A | B_i) \)

\[ P(B_i | A) = \frac{p(B_i) \cdot p(A | B_i)}{\sum_{j=1}^{k} p(B_j) \cdot p(A | B_j)} \]

**ELISA**

\[
\begin{array}{ccc}
\text{HIV-} & \text{ELISA} & \text{antibody concentration} \\
\text{HIV+} & \text{ELISA} & \text{not HIV+} \\
\text{+} & \text{+} & \text{+} \\
\text{-} & \text{-} & \text{-} \\
\end{array}
\]

\[ A = \{ \text{true HIV} \} \]
\[ \text{not A} = \{ \text{true HIV} \} \]

\[ + = \{ \text{ELISA says +} \} \]
\[ - = \{ \text{ELISA says -} \} \]

Sensitivity = 0.96

Specificity = 0.97

\[ P(- | \text{not A}) \]

Prevalence = 0.49%

\[ P(A) = 0.049 \]

\[ \text{prevalence} = 2 \times 0.004 \]
Suppose EMUSA says: for a given blood sample: what \( P(A | +) \)?

Method 1: Bayes's

Then directly, we want \( P(+|A) \) this is how to think about unless we know \( \text{truth}(A, \text{not} A) \) because we do know \( P(+|A) \) and we can work out \( P(-|\text{not}A) \) from that & also \( P(+|\text{not}A) \) \( = 1 - P(-|\text{not}A) = 1 - \text{specificity} < .03 \)
When looking for a partition, Dennis Lindley (1920–2014) suggests:

\[
\text{finding a partition} \quad \Rightarrow \quad \text{"extending the conjunctive"}
\]

A useful partition \( B_1, \ldots, B_k \) has 2 properties: \( P(B_i) > 0 \) & known & \( P(\theta | B_i) \) is also known.

I don't know \( P(+) \) but I do know something about \( P(+ | A) \) and \( P(+) | \text{not } A \) data unknown truth.
\[ p(+1) = p(A) \cdot p(+1|A) + p(\neg A) \cdot p(+1|\neg A) \]

\[ = (0.04)(0.96) + (0.96)(0.03) \]

\[ = 0.0384 + 0.0288 = 0.1112 \]

\[ \therefore p(A|+) = \frac{.0384}{.0372} = 1 \]

\[ p(\neg A|+) = 0.89 \]

\[ p(A|-) = \frac{16}{96628} = 0.0002 = 0.02\% \]

\[ = \text{false negative rate} \]

\[ = \text{false positive rate} \]
method

2 × 2 contingency table

truth

<table>
<thead>
<tr>
<th></th>
<th>has HIV</th>
<th>no HIV</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>384</td>
<td>2,988</td>
</tr>
<tr>
<td>not A</td>
<td>16</td>
<td>96,628</td>
</tr>
</tbody>
</table>

P(A) = prevalence = 0.004

Specificity

utility

EUSA

utility function

optimal decision making

Bayesian decision theory

utility

blood bank

utility

A not A

wasted good unit of blood

(bad) ~ 50-100

blood

somes one gets HIV from bad blood

really bad

optimal decision making

Bayesian decision theory

utility

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posterior odds in favor of $A$

\[
\frac{96}{3} \times \left( \frac{0.04}{0.96} \right) \left( \frac{0.96}{1 - 0.97} \right) \left( \frac{\text{prior odds}}{\text{Bayes factor}} \right) \]

\[
\frac{32}{249} = 0.13 \text{ to } 1 \quad \text{against really HIV+}
\]

\[
1 = \frac{0}{1 + 0} = \frac{32}{249} \approx 0.11
\]

ELISA:

\[
\frac{384}{3372} = 0.11
\]

Fp rate 89%, Fn rate 0.22% (0.5%) / western blot ~ 910.
we want to compare \( P(C_1 | M_2, Y_1) \)

and with \( P(C_3 | M_2, Y_1) \). This is like ELISA:

- unknown \& location of car
- test HIV status
- data:
- monte carlo simulation
- you said behind seat 2
- what ELISA said
- we want \( P(unknown | data) \) but
- problem setup gave us \( P(data | unknown) \)

So let's use Bayes's Theorem to reverse order of conditioning.

**Posterior odds**

\[
\frac{P(C_1 | M_2, Y_1)}{P(C_3 | M_2, Y_1)} = \left[ \frac{P(C_1)}{P(C_3)} \right] P(M_2, Y_1 | C_1) / P(M_2, Y_1 | C_3)
\]

\[P(C_2 | M_2, Y_1) = 0\]
Now by the rules \( P(c_1) = P(c_3) = \frac{1}{3} \), so the prior odds are \( \frac{P(c_1)}{P(c_3)} = \frac{1}{2} = 1 \) to evaluate probabilities like \( P(m_2, y_1 | c_1) \), let's use the general form of product rule for \( \cap \): \[
\frac{P(m_2, y_1 | c_1)}{P(m_2, y_1 | c_3)} = \frac{P(y_1 | c_1) \cdot P(m_2 | y_1, c_1)}{P(y_1 | c_3) \cdot P(m_2 | y_1, c_3)}
\]
but \( y_1 \) and \( c_1 \) are independent so \( P(y_1 | c_1) = P(y_1) = \frac{2}{3} \) and \( P(y_1 | c_3) = P(y_1) = \frac{1}{3} \) so
\[
\frac{P(c_1 | m_2, y_1)}{P(c_3 | m_2, y_1)} = \frac{P(m_2 | y_1, c_1)}{P(m_2 | y_1, c_3)} = \frac{1}{2} = \frac{1}{2}
\]
So after \( m_2 \) (give \( y_2, c_j \)), the posterior odds in favor of car behind door 2 are \( 2:1 \), so \( P(c_3 | m_2, y_2) = \frac{2}{3} \).

For any \( D \) such that \( P(D) > 0 \) and any \( A \),

(a) if \( P(A) = 0 \) then \( P(A|D) = 0 \)

and

(b) if \( P(A) = 1 \) then \( P(A|D) = 1 \).

\( D = \) data \hspace{1cm} P(A) = \) prior information about \( A \)
\( A = \) unknown \hspace{1cm} P(A|D) = \) posterior information about \( A \)
meaning (Anything you put prior probability on has to have posterior probability no matter how the dataset comes out; this destroys the possibility of learning from data.)

\[ P(A | D) = \frac{P(A \text{ and } D)}{P(D)} \]

But if \( P(A) = 0 \) (\( \emptyset \))

then \( P(A \text{ and } D) = 0 \)

\[ P(A | D) = \frac{P(A \text{ and } D)}{P(D)} \]

so \( (A \text{ and } D) = D \)

and \( P(D | A) = 1 \)
Case Study: The Rasmussen Report

WASH-1400, "The Reactor Safety Study"

Problem Estimate $P$ (catastrophic accident at nuclear power plant)
at a moment in history when no such events had ever occurred (nuclear power began in about 1955, ...)

Smile Island 1979

"Solution" Use expert judgment to break down $\Theta$ into a collection of simpler events connected together with $(\text{and}, \; \text{or}, \; ...)$ for example:

$\Theta = (\text{high 1 & alarm & lots break & fail to go off & neglect})$
Result: Estimate of $p(\theta)$ was extremely small: $10^{-12}$, yet only 4 years later: 3 miles.

Right calculation:
$$P(\theta) = P(\text{small}) \cdot P(\text{small}) \cdot P(\text{small})$$

What they did instead: they assumed independence!

$$P(\theta) = P(\text{small}) \cdot P(\text{small}) \cdot P(\text{small})$$

= tiny just because many numbers close to 0 multiplied.