This is dist. of time: functions of RV's, expectation

This is expectation: of sums and products, variance, SD; other moments of the dist.

Real: DS ch. 4 pp. 215 - 260

Case study: option

Case study: pricing

Silicon valley companies give signing bonuses as incentives to accept their job offers. These are often in the form of stock options: an opportunity to buy N shares of the (known) company one year from now at a price. In your view, if the stock is likely to rise over the next year, you'll be able to sell at a profit. Define $X = (\text{price of the stock 1 year from now}) - \text{purchase price}$.

For simplicity, pretend $X$ is discrete
with only 2 values: \( x_1 \leq S \) and \( x_2 > S \). Let \( P = P(X = x_2) \), the prob. that the stock will rise in value. You'd like to evaluate these stock options (e.g., to compare one company's job offer with that of another), but (of course) you don't know \( X \). Let \( Y \) = value of option for one share at \( \$ S \) 1 year from now.

If \( (X = x_1 \leq S) \), the option is worthless and \( Y = 0 \); otherwise (ignoring dividends & costs of buying & selling stocks) if \( (X = x_2 > S) \) then the option is worth \( (x_2 - S) \); thus \( Y = h(X) = \begin{cases} 0 & \text{if } X = x_1 \\ (x_2 - S) & \text{if } X = x_2 \end{cases} \).

To see how valuable the option is, you have to compare...
it to the return you would have received if you had not exercised the stock option; a reasonable point of comparison would be to invest in a bond that pays \( \frac{a}{2} \%/\text{yr.} \)

A fair measure of worth of the option would be the present value of \( E \), defined to be the number \( c \) such that

\[
E(\Xi) = (1 + d) \cdot c.
\]

But we already know that

\[
E(\Xi) = 0 \cdot (1 - p) + (x_2 - S) \cdot p = (x_2 - S) p,
\]

so \((1 + d) \cdot c = (x_2 - S) \cdot p\) and \( c = \left( \frac{x_2 - S}{1 + d} \right) p \).

To finish the calculation, you need to specify \( p \). The standard way to do this...
in the financial sector is to assume that the present value of \( X \) is equal in expectation to the current value of the stock price; i.e., to assume that the expected value of \( (\text{buying 1 share & holding it for a year}) = (\text{investing the same amount of money in the risk-free alternative}) \) — i.e., \( E(X) = (1+\alpha)S \).

But we already know that

\[
E(X) = p \cdot x_2 + (1-p) \cdot x_1 = (1+\alpha)S;
\]

solve for \( p \) gives

\[
p = \frac{x_1 - (1+\alpha)S}{x_1 - x_2} = \frac{(1+\alpha)S - x_1}{x_2 - x_1}.
\]

So the fair price \( c \) of an option to buy
one share is given by:

\[ c = \left( \frac{x_2 - S}{1 + d} \right) \left( \frac{(1 + d)S - x_1}{x_2 - x_1} \right) \]

If we use an illustrative example:
- $S = $200
- $x_1 = $180
- $x_2 = $260
- $d = 0.04$ (realistic)

Downside:
- $x_1 = $180 (-10%)
- $S = $20 (-10%)

Upride:
- $x_2 = $260 (40%)
- $S = $60 (40%)

With these values,
\[ c = \$20.49 \]
(about 10% of the current value of the stock). $c$ is called the risk-neutral price of the option; under the assumptions made here, you could now sell the option today (if you had it) at a fair price of about $\$20$; this would make you an options trader.

An investment that allows people to buy or sell an option on a security is called a derivative (e.g., stock).
Examples of invertible (1-1) & differentiable functions.

\[ f(x) = e^x \quad \text{and} \quad f(x) = \ln(x) \]

\[ f(x) = x^2 \quad \text{diff. but not} \quad 1-1 \]

50% 50% 
point of symmetry = mode = median = mean

Symmetric & unimodal

exponential family: income in us.

$0 < \text{mode} < \text{mean} < \text{median}$
mean is pulled by the tail

mean much more influenced by outliers than median

house prices in SC

unsurprising real estate person quote median to a buyer but mean to a seller

you: ask for both mean & median
\[ f_{X}(x) \]
\[ Y = X + 10 \]
\[ E(Y) = E(X) + 10 \]
\[ f_{Y}(y) \]
\[ Y = 2X \]
\[ E(Y) = 2E(X) \]
\[ Y = 9X + 6 \rightarrow E(Y) = 9E(X) + 6 \]
\[ f_{X}(x) \]
\[ Y = -X \]
\[ (9.55, 10, 40) \]

Plan Ahead
Some spreading & shape different

$E(X)$

Same center & shape, different spread

Same center & spread

Different shape
\( X_1, X_2 \text{ independent} \)

\[
\begin{align*}
\text{SD}(X_1) &= 2 \\
\text{SD}(X_2) &= 7
\end{align*}
\]

\[
\text{SD}(X_1 + X_2) = \star
\]

\[
\text{SD}(X_1 + X_2) = 11
\]

\[
\text{Empirical Rule of a Normal Distribution.}
\]

For (virtually) any distribution, if you start at the mean, \( \mu_X = E(X) \) and

\[
\left\{ \frac{1}{2}, \frac{2}{3} \right\} \text{ SDs, } \sigma_X = \sqrt{\text{Var}(X)} \text{ either way, you will usually capture}
\]
\[
\begin{align*}
\{ & \text{about } \frac{1}{3} \ (68\%) \} \\
& \text{most } \ (95\%) \\
& \text{almost all } \ (99.7\%) \\
\end{align*}
\]

Probability

\[
\begin{align*}
\text{SD} & = 2.5 \\
f_\chi^2(x) & \quad \text{for } x \in (49, 51) \\
E(\chi^2) & = 50 \\
\text{SD}(\chi^2) & = \sigma^2 = x \\
& \in (-25, 125) \\
\end{align*}
\]

\[
\begin{align*}
0 = 50 + 2: \text{ likelihood} \\
\text{center} = \text{pt. of symmetry} \\
\text{standard Cauchy} \\
\end{align*}
\]

Free Symbolic: Wolfram alpha
Symmetric

Skewness 0

Positively skewed (skew > 0)

Negatively skewed (skew < 0)
\[ f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{else} \end{cases} \]

\[ E(X) = \frac{1}{\lambda} \]

(positive skew)

\[ \chi \]

\[ 0 \]

\[ 1 \]

\[ e^x \]

\[ 1 \]

\[ 0.1 \]

\[ 0 \]

\[ 1.0 \]

\[ x \]

\[ 0.1 \]

\[ \eta = 0.1 \]

\[ \eta = 1 \]

Less skewed

More skewed